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A note on logarithms of self-adjoint operators

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Throughout this note $\mathcal{H}$ will denote a complex Hilbert space, $\mathcal{L}(\mathcal{H})$ the set of all bounded linear operators on $\mathcal{H}$, endowed with the usual structure of a Banach space, $\sigma(T)$ and $r(T)$ will denote the spectrum of $T \in \mathcal{L}(\mathcal{H})$ and the spectral radius of T, respectively.

In [4], C.R. Putnam has proved that if A is a positive self-adjoint operator in $\mathcal{L}(\mathcal{H})$, $T \in \mathcal{L}(\mathcal{H})$ and $e^T = A$, then $\|T\| \leq 2\log 2$ implies that T is self-adjoint. In [2], S. Kurepa has shown that it is sufficient to assume that $\|T\| < 2\pi$ in order that T be self-adjoint. This condition, already in the set of complex numbers, cannot be replaced by $\|T\| \leq 2\pi$ without changing the conclusion.

The object of the present note is to give a new proof of Kurepa's result. Furthermore we will generalize some of the results in [2]. To this end we will use the following propositions.

Proposition 1 *Suppose that $T \in \mathcal{L}(\mathcal{H})$ is normal. Then:*

(a) $r(T) = \|T\|$,

(b) *T is self-adjoint if and only if $\sigma(T) \subseteq \mathbb{R}$.*

Proof. (a) is shown in [3, Lemma 4.3.11] and (b) is shown in [3, Proposition 4.4.7]. ■

A set $\Omega \subset \mathbb{C}$ is called $2\pi i$-congruence-free, if $\lambda_1, \lambda_2 \in \Omega$ and $\lambda_1 \equiv \lambda_2 \pmod{2\pi i}$ imply that $\lambda_1 = \lambda_2$.

The following result is due to E. Hille, [1].

Proposition 2 *Let $T, S \in \mathcal{L}(\mathcal{H})$, let $\sigma(T)$ be $2\pi i$-congruence-free and let*

$$e^T = e^S.$$

Then $TS = ST$.

Theorem 1 *If A and T are operators in $\mathcal{L}(\mathcal{H})$, A is positive and self-adjoint,*

$$e^T = A \quad \text{and} \quad r(T) < 2\pi,$$

then T is self-adjoint.

Proof. Since A is positive and self-adjoint and $A = e^T$, we have $\sigma(A) \subseteq (0, \infty)$. Now take $\lambda \in \sigma(T)$. Then $e^\lambda \in \sigma(A)$, thus $e^\lambda \in (0, \infty)$. Hence there is $\alpha \in \mathbb{R}$ and $k \in \mathbb{Z}$ such that $\lambda = \alpha + 2k\pi i$. It follows that $|\lambda|^2 = \alpha^2 + 4k^2\pi^2 < 4\pi^2$, so that $k = 0$, thus $\lambda = \alpha \in \mathbb{R}$. This shows that $\sigma(T) \subseteq \mathbb{R}$ and therefore $\sigma(T)$ is $2\pi i$-congruence-free. From

$$e^{T^*} = (e^T)^* = A^* = A = e^T$$

and Proposition 2 we get that T is normal. Proposition 1(b) shows now that T is self-adjoint. ■

As mentioned in the introduction, the condition $r(T) < 2\pi$ cannot be replaced by $r(T) \leq 2\pi$. But we have

Theorem 2 *Suppose that $A, T \in \mathcal{L}(\mathcal{H})$, A is positive and self-adjoint,*

$$e^T = A, \quad r(T) \leq 2\pi \quad \text{and} \quad 2\pi i, -2\pi i \notin \sigma(T),$$

then T is self-adjoint.

Proof. Take $\lambda \in \sigma(T)$. As in the proof of Theorem 1, $\lambda = \alpha + 2k\pi i$ for some $\alpha \in \mathbb{R}$ and some $k \in \mathbb{Z}$. From $|\lambda|^2 = \alpha^2 + 4k^2\pi^2 \leq 4\pi^2$, we see that $k \in \{0, 1, -1\}$. If $k = \pm 1$ then $\alpha = 0$ and therefore $\lambda = \pm 2\pi i$. But this is a contradiction, since $\pm 2\pi i \notin \sigma(T)$. It follows that $\sigma(T) \subseteq \mathbb{R}$. As in the proof of Theorem 1 we see that T is self-adjoint. ■

Corollary 1 *If T and S are operators in $\mathcal{L}(\mathcal{H})$, S is self-adjoint,*

$$e^T = e^S \quad \text{and} \quad r(T) < 2\pi,$$

then $T = S$.

Proof. Put $A = e^S$. Then A is self-adjoint. By $(\cdot|\cdot)$ we denote the inner product on $\mathcal{H}$. Since

$$(Ax|x) = (e^{S/2}e^{S/2}x|x) = (e^{S/2}x|e^{S/2}x) = \|e^{S/2}x\|^2 \geq 0$$

for each $x \in \mathcal{H}$, A is positive. From Theorem 1 we conclude that T is self-adjoint. Proposition 2 gives $TS = ST$, thus $e^{T-S} = I$. Now take $\lambda \in \sigma(T - S)$. Then $e^\lambda = 1$. Since $T - S$ is self-adjoint, $\lambda \in \mathbb{R}$. Hence $\lambda = 0$. Therefore $\sigma(T - S) = \{0\}$. Use Proposition 1(a) to derive $\|T - S\| = r(T - S) = 0$. Hence $T = S$, as desired. ■

Corollary 2 *If $T, S \in \mathcal{L}(\mathcal{H})$, S is self-adjoint,*

$$e^T = e^S, \quad r(T) \leq 2\pi \quad \text{and} \quad 2\pi i, -2\pi i \notin \sigma(T),$$

then $T = S$.

Proof. Argue as in the proof of Corollary 1. Use Theorem 2 to see that T is self-adjoint. ■

The following corollary can be found in [2]. We will give a slightly different proof.

Corollary 3 *Let $T, A \in \mathcal{L}(\mathcal{H})$ and $\theta \in [0, 2\pi]$. Suppose that A is positive and self-adjoint and that $e^T = e^{i\theta}A$.*

(a) *If* $\theta \in [0, \pi]$, *then* $r(T) \geq \theta$.

(b) *If* $\theta \in [\pi, 2\pi]$, *then* $r(T) \geq 2\pi - \theta$.

Proof. (a) Suppose that $r(T) < \theta$. Then

$$r(T - i\theta I) \leq r(T) + \theta < 2\theta < 2\pi.$$

From $e^{T - i\theta I} = e^T e^{-i\theta} I = A$ and Theorem 1, we see that $T - i\theta I$ is self-adjoint, thus T is normal and $T - T^* = 2i\theta$. Since T and T^* commute, $r(T - T^*) \leq r(T) + r(T^*)$ (see [3, Exercise 4.1.12]). Thus

$$2\theta = r(T - T^*) \leq r(T) + r(T^*) = 2r(T) < 2\theta,$$

a contradiction.
(b) Put $\tau = 2\pi - \theta$. Then $e^{T^*} = e^{-i\theta} A = e^{i(2\pi - \theta)} A = e^{i\tau} A$. Since $\tau \in [0, \pi]$, (a) shows that $r(T^*) \geq \tau$. Thus $r(T) \geq 2\pi - \theta$. ■

As an immediate consequence of Corollary 3 we have:

Corollary 4 *Suppose that* $T, A \in \mathcal{L}(\mathcal{H})$ *and that* A *is positive and self-adjoint.*

(a) *If* $e^T = -A$, *then* $r(T) \geq \pi$.

(b) *If* $e^T = iA$, *then* $r(T) \geq \frac{\pi}{2}$.

(c) *If* $e^T = -iA$, *then* $r(T) \geq \frac{\pi}{2}$.

We close this paper with results concerning logarithms of unitary operators.

Theorem 3 *Suppose that* $U \in \mathcal{L}(\mathcal{H})$ *is unitary,* $T \in \mathcal{L}(\mathcal{H})$, $r(T) \leq \pi$ *and* $e^{iT} = U$. *If* $\pi \notin \sigma(T)$ *or* $-\pi \notin \sigma(T)$, *then* T *is self-adjoint.*

Proof. Take $\lambda \in \sigma(iT)$. Then $e^\lambda \in \sigma(U)$, thus $|e^\lambda| = 1$. Hence $\lambda = i\beta$ for some $\beta \in \mathbb{R}$. From $|\beta| = |\lambda| \leq r(T) \leq \pi$ we see that

$$\sigma(iT) \subseteq \{i\beta : \beta \in [-\pi, \pi]\}.$$

Since $\pi \notin \sigma(T)$ or $-\pi \notin \sigma(T)$, $\sigma(iT)$ is $2\pi i$-congruence-free. From

$$e^{-iT^*} = (e^{iT})^* = U^* = U^{-1} = e^{-iT}$$

and Proposition 2 we derive $TT^* = T^*T$. Hence T is normal. Furthermore we have $\sigma(T) \subseteq [-\pi, \pi]$. It follows from Proposition 1(b) that T is self-adjoint. ■

Corollary 5 *If* $T \in \mathcal{L}(\mathcal{H})$, $r(T) < \pi$ *and* e^{iT} *is unitary, then* T *is self-adjoint.*

References

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